

Elementary Derivation of the Dirac Equation. VIII

Hans Sallhofer

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The momentum balance of Dirac-like electrodynamics is established in Da Silveira's notation.

The notation of De Silveira [1] obviously is of great heuristic importance. In establishing the Maxwell-Dirac isomorphy, for instance, it avoids the artifice [2, (6)] and, in addition, brings about the Darwin relation [2, (7)] automatically. Therefore we want to investigate how the momentum balance [4, (11)] is represented in Da Silveira's formalism.

According to De Silveira we may put the Dirac-like electrodynamics [2, (5)] into the form

$$\left[\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \cdot \nabla - \begin{pmatrix} 0 & \mu \mathbf{1} \\ \varepsilon \mathbf{1} & 0 \end{pmatrix} \frac{\partial}{\partial (ct)} \right] \begin{bmatrix} i(\sigma \cdot E) \\ (\sigma \cdot H) \end{bmatrix} = 0. \quad (1)$$

Multiplication by the matrix

$$\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad (2)$$

from the left yields

$$\left[\begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \cdot \nabla - \begin{pmatrix} \varepsilon \mathbf{1} & 0 \\ 0 & \mu \mathbf{1} \end{pmatrix} \frac{\partial}{\partial (ct)} \right] \begin{bmatrix} i(\sigma \cdot E) \\ (\sigma \cdot H) \end{bmatrix} = 0. \quad (3)$$

If we identify, as in [2, (7)] and [2, (8)],

$$\gamma^{(j)} = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \gamma^{(k)} = - \begin{pmatrix} \varepsilon \mathbf{1} & 0 \\ 0 & \mu \mathbf{1} \end{pmatrix},$$

$$\Psi = \begin{bmatrix} i(\sigma \cdot E) \\ (\sigma \cdot H) \end{bmatrix}, \quad (4)$$

then (3) is identical with [2, (8)]:

$$[\gamma \cdot \nabla + \gamma^{(k)} \partial_k] \Psi = 0. \quad (5)$$

The wavefunction now is a matrix with four rows and two columns:

$$\Psi = \begin{bmatrix} i(\sigma \cdot E) \\ (\sigma \cdot H) \end{bmatrix} = \begin{pmatrix} iE_3 & i(E_1 - iE_2) \\ i(E_1 + iE_2) & -iE_3 \\ H_3 & H_1 - iH_2 \\ H_1 + iH_2 & -H_3 \end{pmatrix}, \quad (6)$$

Reprint requests to Dr. Hans Sallhofer, Fischer-Strasse 12, A-5280 Braunau, Austria.

where [2, (7)] is represented by the first column.

In order to obtain the energy relations we repeat the procedure leading to [3, (5)]. For this we take (5) and its hermitian conjugate. They read explicitly

$$\gamma \cdot (\nabla \Psi) + \frac{1}{c} \gamma_k \left(\frac{\partial}{\partial t} \Psi \right) = 0, \quad (7)$$

$$(\nabla \Psi^{kc}) \cdot \gamma + \frac{1}{c} \left(\frac{\partial}{\partial t} \Psi^{kc} \right) \gamma_k = 0, \quad (8)$$

where Ψ^{kc} has the form

$$\Psi^{kc} = [-i(\sigma \cdot E), (\sigma \cdot H^*)] \quad (9)$$

$$= \begin{bmatrix} -iE_3^* & -i(E_1^* - iE_2^*) & H_3^* & H_1^* - iH_2^* \\ -i(E_1^* + iE_2^*) & iE_3^* & H_1^* + iH_2^* & -H_3^* \end{bmatrix}.$$

Multiplication of (7) by Ψ^{kc} from the left, (8) by Ψ from the right, and subsequent addition yields

$$\frac{1}{c} \frac{\partial}{\partial t} (\Psi^{kc} \gamma_k \Psi) + \text{div} (\Psi^{kc} \gamma \Psi) = 0. \quad (10)$$

The two terms are

$$\Psi^{kc} \gamma_4 \Psi = -[-i(\sigma \cdot E^*), (\sigma \cdot H^*)] \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{bmatrix} i(\sigma \cdot E) \\ (\sigma \cdot H) \end{bmatrix}$$

$$= -\varepsilon(\sigma \cdot E^*)(\sigma \cdot E) - \mu(\sigma \cdot H^*)(\sigma \cdot H) \quad (11)$$

and

$$\Psi^{kc} \gamma \Psi = [-i(\sigma \cdot E^*), (\sigma \cdot H^*)] \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \begin{bmatrix} i(\sigma \cdot E) \\ (\sigma \cdot H) \end{bmatrix}$$

$$= i[(\sigma \cdot H^*) \sigma (\sigma \cdot E) - (\sigma \cdot E^*) \sigma (\sigma \cdot H)]. \quad (12)$$

With the help of the two relations

$$(\sigma \cdot A)(\sigma \cdot B) = (A \cdot B) + i\sigma \times (A \times B) \quad (13)$$

and

$$(\sigma \cdot A) \sigma (\sigma \cdot B) = A(\sigma \cdot B) + (\sigma \cdot A) B - i(A \times B) - (A \cdot B) \sigma, \quad (14)$$

we get

$$\Psi^{kc} \gamma_k \Psi = -\varepsilon[(E^{\text{Re}})^2 + (E^{\text{Im}})^2] - \mu[(H^{\text{Re}})^2 + (H^{\text{Im}})^2]$$

$$- 2\varepsilon \sigma \cdot (E^{\text{Im}} \times E^{\text{Re}}) - 2\mu \sigma \cdot (H^{\text{Im}} \times H^{\text{Re}}) \quad (15)$$

and

$$\Psi^{kc} \gamma \Psi = -2[(E^{\text{Re}} \times H^{\text{Re}}) + (E^{\text{Im}} \times H^{\text{Im}})$$

$$+ H^{\text{Re}}(\sigma \cdot E^{\text{Im}}) - H^{\text{Im}}(\sigma \cdot E^{\text{Re}})$$

$$+ E^{\text{Im}}(\sigma \cdot H^{\text{Re}}) - E^{\text{Re}}(\sigma \cdot H^{\text{Im}})$$

$$+ (E^{\text{Re}} \cdot H^{\text{Im}}) \sigma - (E^{\text{Im}} \cdot H^{\text{Re}}) \sigma]. \quad (16)$$

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Thus (10) becomes

$$\begin{aligned}
 & \left\{ \frac{1}{c} \frac{\partial}{\partial t} \frac{1}{2} [\varepsilon (\mathbf{E}^{\text{Re}})^2 + \mu (\mathbf{H}^{\text{Re}})^2] + \text{div} (\mathbf{E}^{\text{Re}} \times \mathbf{H}^{\text{Re}}) \right\} \\
 & + \left\{ \frac{1}{c} \frac{\partial}{\partial t} \frac{1}{2} [\varepsilon (\mathbf{E}^{\text{Im}})^2 + \mu (\mathbf{H}^{\text{Im}})^2] + \text{div} (\mathbf{E}^{\text{Im}} \times \mathbf{H}^{\text{Im}}) \right\} \\
 & + \frac{1}{c} \frac{\partial}{\partial t} [\varepsilon (\mathbf{E}^{\text{Im}} \times \mathbf{E}^{\text{Re}}) + \mu (\mathbf{H}^{\text{Im}} \times \mathbf{H}^{\text{Re}})] \cdot \boldsymbol{\sigma} \quad (17) \\
 & + \text{div} \{ \mathbf{H}^{\text{Re}} (\mathbf{E}^{\text{Im}} \cdot \boldsymbol{\sigma}) - \mathbf{H}^{\text{Im}} (\mathbf{E}^{\text{Re}} \cdot \boldsymbol{\sigma}) + \mathbf{E}^{\text{Im}} (\mathbf{H}^{\text{Re}} \cdot \boldsymbol{\sigma}) \\
 & - \mathbf{E}^{\text{Re}} (\mathbf{H}^{\text{Im}} \cdot \boldsymbol{\sigma}) - [(\mathbf{E}^{\text{Im}} \cdot \mathbf{H}^{\text{Re}}) - (\mathbf{E}^{\text{Re}} \cdot \mathbf{H}^{\text{Im}})] \boldsymbol{\sigma} \} = 0.
 \end{aligned}$$

The first two terms in curly brackets vanish for any charge-free electrodynamics according to [3, (1)]. Using [4, (5)] and [4, (6)] the remainder can be put into the form

$$(\dot{V} = \text{div } \mathbf{T}) \cdot \boldsymbol{\sigma} = 0, \quad (18)$$

or more generally

$$\dot{V} + \text{div } \mathbf{T} = 0, \quad (19)$$

in agreement with [4, (11)].

The above derivation of the momentum balance suggests, among others, that the structure of Dirac-like electrodynamics is closely related to the Pauli matrices.

[1] Da Silveira, Z. Naturforsch. **34a**, 646 (1979).

[2] H. Sallhofer, Z. Naturforsch. **33a**, 1378 (1978).

[3] H. Sallhofer, Z. Naturforsch. **34a**, 1145 (1979).

[4] H. Sallhofer, Z. Naturforsch. **39a**, 1142 (1984).