Elementary Derivation of the Dirac Equation. VIII

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The momentum balance of Dirac-like electrodynamics is established in Da Silveira's notation.

The notation of De Silveira [1] obviously is of great heuristic importance. In establishing the Maxwell-Dirac isomorphy, for instance, it avoids the artifice [2, (6)] and, in addition, brings about the Darwin relation [2, (7)] automatically. Therefore we want to investigate how the momentum balance [4, (11)] is represented in Da Silveira's formalism.

According to De Silveira we may put the Dirac-like electrodynamics [2, (5)] into the form

$$\left[\begin{pmatrix}\boldsymbol{\sigma} & \mathbb{0}\\ \mathbb{0} & \boldsymbol{\sigma}\end{pmatrix}\cdot\nabla-\begin{pmatrix}\mathbb{0} & \mu\mathbb{1}\\ \varepsilon\mathbb{1} & \mathbb{0}\end{pmatrix}\frac{\partial}{\partial(ct)}\right]\begin{bmatrix}i(\boldsymbol{\sigma}\cdot\boldsymbol{E})\\ (\boldsymbol{\sigma}\cdot\boldsymbol{H})\end{bmatrix}=0. \quad (1)$$

Multiplication by the matrix

$$\begin{pmatrix}\mathbb{0} & \mathbb{1}\\ \mathbb{1} & \mathbb{0}\end{pmatrix}. \quad (2)$$

from the left yields

$$\left[\begin{pmatrix}\mathbb{0} & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & \mathbb{0}\end{pmatrix}\cdot\nabla-\begin{pmatrix}\varepsilon\mathbb{1} & \mathbb{0}\\ \mathbb{0} & \mu\mathbb{1}\end{pmatrix}\frac{\partial}{\partial(ct)}\right]\begin{bmatrix}i(\boldsymbol{\sigma}\cdot\boldsymbol{E})\\ (\boldsymbol{\sigma}\cdot\boldsymbol{H})\end{bmatrix}=0. \quad (3)$$

If we identify, as in [2, (7)] and [2, (8)],

$$\gamma^{(j)}=\begin{pmatrix}\mathbb{0} & \boldsymbol{\sigma}_j\\ \boldsymbol{\sigma}_j & \mathbb{0}\end{pmatrix},\quad \gamma^{(k)}=-\begin{pmatrix}\varepsilon\mathbb{1} & \mathbb{0}\\ \mathbb{0} & \mu\mathbb{1}\end{pmatrix},$$

$$\Psi=\begin{bmatrix}i(\boldsymbol{\sigma}\cdot\boldsymbol{E})\\ (\boldsymbol{\sigma}\cdot\boldsymbol{H})\end{bmatrix}, \quad (4)$$

then (3) is identical with [2, (8)]:

$$[\gamma\cdot\nabla+\gamma^{(k)}\,\partial_k]\,\Psi=0. \quad (5)$$

The wavefunction now is a matrix with four rows and two columns:

$$\Psi=\begin{bmatrix}i(\boldsymbol{\sigma}\cdot\boldsymbol{E})\\ (\boldsymbol{\sigma}\cdot\boldsymbol{H})\end{bmatrix}=\begin{matrix}iE_3 & i(E_1-iE_2)\\ i(E_1+iE_2) & -iE_3\\ H_3 & H_1-iH_2\\ H_1+iH_2 & -H_3\end{matrix}, \quad (6)$$

where [2, (7)] is represented by the first column.

In order to obtain the energy relations we repeat the procedure leading to [3, (5)]. For this we take (5) and its hermitian conjugate. They read explicitly

$$\gamma\cdot(\nabla\Psi)+\frac{1}{c}\gamma_k\left(\frac{\partial}{\partial t}\Psi\right)=0, \quad (7)$$

$$(\nabla\Psi^{kc})\cdot\gamma+\frac{1}{c}\left(\frac{\partial}{\partial t}\Psi^{kc}\right)\gamma_k=0, \quad (8)$$

where Ψ^{kc} has the form

$$\Psi^{kc}=[-i(\boldsymbol{\sigma}\cdot\boldsymbol{E}),(\boldsymbol{\sigma}\cdot\boldsymbol{H}^*)] \quad (9)$$

$$=\begin{bmatrix}-iE_3^* & -i(E_1^*-iE_2^*) & H_3^* & H_1^*-iH_2^*\\ -i(E_1^*+iE_2^*) & iE_3^* & H_1^*+iH_2^* & -H_3^*\end{bmatrix}.$$

Multiplication of (7) by Ψ^{kc} from the left, (8) by Ψ from the right, and subsequent addition yields

$$\frac{1}{c}\frac{\partial}{\partial t}(\Psi^{kc}\gamma_k\Psi)+\operatorname{div}(\Psi^{kc}\gamma\Psi)=0. \quad (10)$$

The two terms are

$$\Psi^{kc}\gamma_4\Psi=-[-i(\boldsymbol{\sigma}\cdot\boldsymbol{E}^*),(\boldsymbol{\sigma}\cdot\boldsymbol{H}^*)]\begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix}\begin{bmatrix}i(\boldsymbol{\sigma}\cdot\boldsymbol{E})\\ (\boldsymbol{\sigma}\cdot\boldsymbol{H})\end{bmatrix}$$

$$=-\varepsilon(\boldsymbol{\sigma}\cdot\boldsymbol{E}^*)(\boldsymbol{\sigma}\cdot\boldsymbol{E})-\mu(\boldsymbol{\sigma}\cdot\boldsymbol{H}^*)(\boldsymbol{\sigma}\cdot\boldsymbol{H}) \quad (11)$$

and

$$\Psi^{kc}\gamma\Psi=[-i(\boldsymbol{\sigma}\cdot\boldsymbol{E}^*),(\boldsymbol{\sigma}\cdot\boldsymbol{H}^*)]\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}\begin{bmatrix}i(\boldsymbol{\sigma}\cdot\boldsymbol{E})\\ (\boldsymbol{\sigma}\cdot\boldsymbol{H})\end{bmatrix}$$

$$=i[(\boldsymbol{\sigma}\cdot\boldsymbol{H}^*)\,\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\boldsymbol{E})-(\boldsymbol{\sigma}\cdot\boldsymbol{E}^*)\,\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\boldsymbol{H})]. \quad (12)$$

With the help of the two relations

$$(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B})=(\boldsymbol{A}\cdot\boldsymbol{B})+i\boldsymbol{\sigma}\times(\boldsymbol{A}\times\boldsymbol{B}) \quad (13)$$

and

$$(\boldsymbol{\sigma}\cdot\boldsymbol{A})\,\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\boldsymbol{B}) \quad (14)$$

$$=\boldsymbol{A}(\boldsymbol{\sigma}\cdot\boldsymbol{B})+(\boldsymbol{\sigma}\cdot\boldsymbol{A})\,\boldsymbol{B}-i(\boldsymbol{A}\times\boldsymbol{B})-(\boldsymbol{A}\cdot\boldsymbol{B})\,\boldsymbol{\sigma},$$

we get

$$\Psi^{kc}\gamma_k\Psi=-\varepsilon[(\boldsymbol{E}^{\mathrm{Re}})^2+(\boldsymbol{E}^{\mathrm{Im}})^2]-\mu[(\boldsymbol{H}^{\mathrm{Re}})^2+(\boldsymbol{H}^{\mathrm{Im}})^2]$$
$$-2\varepsilon\,\boldsymbol{\sigma}\cdot(\boldsymbol{E}^{\mathrm{Im}}\times\boldsymbol{E}^{\mathrm{Re}})-2\mu\,\boldsymbol{\sigma}\cdot(\boldsymbol{H}^{\mathrm{Im}}\times\boldsymbol{H}^{\mathrm{Re}})$$

and (15)

$$\Psi^{kc}\gamma\Psi=-2[(\boldsymbol{E}^{\mathrm{Re}}\times\boldsymbol{H}^{\mathrm{Re}})+(\boldsymbol{E}^{\mathrm{Im}}\times\boldsymbol{H}^{\mathrm{Im}})$$
$$+\boldsymbol{H}^{\mathrm{Re}}(\boldsymbol{\sigma}\cdot\boldsymbol{E}^{\mathrm{Im}})-\boldsymbol{H}^{\mathrm{Im}}(\boldsymbol{\sigma}\cdot\boldsymbol{E}^{\mathrm{Re}})$$
$$+\boldsymbol{E}^{\mathrm{Im}}(\boldsymbol{\sigma}\cdot\boldsymbol{H}^{\mathrm{Re}})-\boldsymbol{E}^{\mathrm{Re}}(\boldsymbol{\sigma}\cdot\boldsymbol{H}^{\mathrm{Im}})$$
$$+(\boldsymbol{E}^{\mathrm{Re}}\cdot\boldsymbol{H}^{\mathrm{Im}})\,\boldsymbol{\sigma}-(\boldsymbol{E}^{\mathrm{Im}}\cdot\boldsymbol{H}^{\mathrm{Re}})\,\boldsymbol{\sigma}]. \quad (16)$$

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Thus (10) becomes

$$\left\{\frac{1}{c}\frac{\partial}{\partial t}\frac{1}{2}\left[\varepsilon(\boldsymbol{E}^{\mathrm{Re}})^2+\mu(\boldsymbol{H}^{\mathrm{Re}})^2\right]+\operatorname{div}(\boldsymbol{E}^{\mathrm{Re}}\times\boldsymbol{H}^{\mathrm{Re}})\right\}$$
$$+\left\{\frac{1}{c}\frac{\partial}{\partial t}\frac{1}{2}\left[\varepsilon(\boldsymbol{E}^{\mathrm{Im}})^2+\mu(\boldsymbol{H}^{\mathrm{Im}})^2\right]+\operatorname{div}(\boldsymbol{E}^{\mathrm{Im}}\times\boldsymbol{H}^{\mathrm{Im}})\right\}$$
$$+\frac{1}{c}\frac{\partial}{\partial t}\left[\varepsilon(\boldsymbol{E}^{\mathrm{Im}}\times\boldsymbol{E}^{\mathrm{Re}})+\mu(\boldsymbol{H}^{\mathrm{Im}}\times\boldsymbol{H}^{\mathrm{Re}})\right]\cdot\boldsymbol{\sigma} \tag{17}$$
$$+\operatorname{div}\{\boldsymbol{H}^{\mathrm{Re}}(\boldsymbol{E}^{\mathrm{Im}}\cdot\boldsymbol{\sigma})-\boldsymbol{H}^{\mathrm{Im}}(\boldsymbol{E}^{\mathrm{Re}}\cdot\boldsymbol{\sigma})+\boldsymbol{E}^{\mathrm{Im}}(\boldsymbol{H}^{\mathrm{Re}}\cdot\boldsymbol{\sigma})$$
$$-\boldsymbol{E}^{\mathrm{Re}}(\boldsymbol{H}^{\mathrm{Im}}\cdot\boldsymbol{\sigma})-[(\boldsymbol{E}^{\mathrm{Im}}\cdot\boldsymbol{H}^{\mathrm{Re}})-(\boldsymbol{E}^{\mathrm{Re}}\cdot\boldsymbol{H}^{\mathrm{Im}})]\,\boldsymbol{\sigma}\}=0.$$

The first two terms in curly brackets vanish for any charge-free electrodynamics according to [3, (1)]. Using [4, (5)] and [4, (6)] the remainder can be put into the form

$$(\dot{\boldsymbol{V}}=\operatorname{div}\mathbf{T})\cdot\boldsymbol{\sigma}=0, \tag{18}$$

or more generally

$$\dot{\boldsymbol{V}}+\operatorname{div}\mathbf{T}=0, \tag{19}$$

in agreement with [4, (11)].

The above derivation of the momentum balance suggests, among others, that the structure of Dirac-like electrodynamics is closely related to the Pauli matrices.

[1] Da Silveira, Z. Naturforsch. **34a,** 646 (1979).
[2] H. Sallhofer, Z. Naturforsch. **33a,** 1378 (1978).
[3] H. Sallhofer, Z. Naturforsch. **34a,** 1145 (1979).
[4] H. Sallhofer, Z. Naturforsch. **39a,** 1142 (1984).